SIMILAR SOLUTIONS IN THE THEORY OF NONSTEADY BURNING OF A SOLID PROPELLANT

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The similar solution of the problem of the nonsteady burning rate of a solid propellant for falling pressure, obtained by Zel'dovich for a particular dependence of the steady-state burning rate on pressure and initial temperature [1], is extended to the case of an arbitrary dependence. It is shown that under rather general assumptions two solutions, corresponding to low and high burning rates, may exist. In the presence of a sharp change of pressure the similar solutions cease to exist, which can be related to the attainment of critical quenching conditions. The integral equation relating the burning rate, surface temperature and temperature gradient at the surface, derived in [2], is used to obtain similar solutions for the model with variable surface temperature. It is shown that these solutions can exist only if certain relations between the surface temperature and the pressure and the temperature gradient at the surface are satisfied. An approximate relation between the arbitrary parameters of the solution and the kinetic characteristics of the propellant gasification reaction is established. The stability of the similar solutions at constant surface temperature is investigated; of the two possible solutions only the one corresponding to a lower burning rate is found to be stable.

1. In [1] a solution was obtained for the nonlinear problem of the nonsteady burning rate of a propellant in the case when the steady-state burning rate u° is related as follows with the pressure p and the initial temperature of the propellant T_0

$$u^\circ = Bp^\nu e^{\beta T_0} \qquad (B,\ \nu,\beta = \text{const},\ \nu < 1,\ B > 0) \tag{1.1}$$

It was found that the pressure should decrease with time according to the law

$$p = At^{-\frac{1}{2\nu}} \qquad (A = \text{const} > 0) \tag{1.2}$$

We will consider the case of an arbitrary $u^\circ(p, T_0)$ dependence, assuming, as in [1], that the surface temperature of the propellant is constant. We start from the heat conduction equation describing the one-dimensional propagation of heat in the propellant

$$\frac{\partial\theta}{\partial\tau} + w\frac{\partial\theta}{\partial\xi} = \frac{\partial^2\theta}{\partial\xi^2}, \qquad \begin{matrix} -\infty < \xi \leqslant 0 \\ -\infty < \tau < \infty \end{matrix}$$

$$\theta = \frac{T - T_0}{T_S - T_0}, \quad w = \frac{u}{U}, \quad \tau = \frac{u^2 t}{\varkappa}, \quad \xi = \frac{Ux}{\varkappa} \tag{1.3}$$

Here, T, T_S, and T_0 are the variable temperature, surface temperature, and initial temperature; u is the linear burning rate; U is the characteristic rate (to be defined later); $\varkappa$, t, and x are the thermal diffusivity, time, and the space coordinate (the coordinate origin is tied to the surface of the propellant).

It is known [3] that Eq. (1.3) with boundary conditions

$$\xi = -\infty, \ \ \theta = 0; \ \xi = 0, \theta = 1 \tag{1.4}$$

has a similar solution if

$$w = C\tau^{-1/2} \qquad (C = \text{const} > 0) \tag{1.5}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 34-43, July-August, 1971. Original article submitted October 13, 1970.

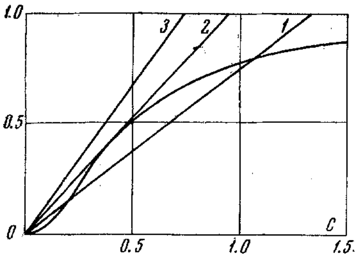

Fig. 1

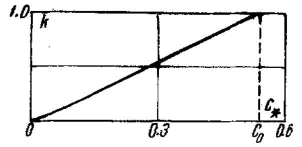

Fig. 2

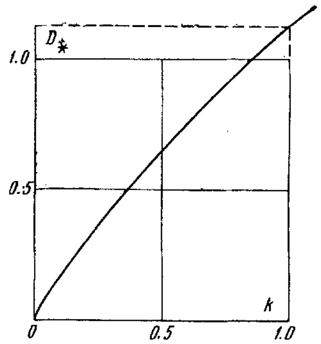

Fig. 3

This solution has the form

$$\theta = \frac{\operatorname{erfc}(C-y)}{\operatorname{erfc} C} \qquad \left(\operatorname{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-y^2}dy, \quad y = \frac{\xi}{2\sqrt{\tau}}\right) \tag{1.6}$$

The similar solution may be regarded as simply an intermediate asymptotic form of the solution of the problem of transition from one steady-state burning regime to another, if on a certain interval of the transient process the law of pressure variation gives a time dependence of the burning rate of the form (1.5). Intermediate means that the system has ceased to "recall" the initial conditions of steady-state burning, but is still far from the final steady-state regime, when the pressure variation deviates from that corresponding to the similar solution.

In order to determine the law of pressure variation with time that gives a burning rate of the form (1.5), we assume that the function

$$w = w(p, \varphi) \qquad (\varphi = (\partial\theta / \partial\xi)_S) \tag{1.7}$$

is known, since it can be obtained [1] from the experimentally determined dependence of the burning rate on pressure and initial temperature in the steady-state regime $u° = u°(p, T_0)$. Substituting in this relation the expression for the initial temperature in terms of the surface temperature and the temperature gradient at the surface, we obtain

$$u = u\left[p,\ T_S - \frac{\varkappa}{u}\left(\frac{\partial T}{\partial x}\right)_S\right] \tag{1.8}$$

which is also correct for the nonsteady regime [1, 4].

For similar solution (1.6) we have

$$w = \frac{C}{\sqrt{\tau}} = w(p(\tau),\ T_*) \qquad \left(T_* = T_S - \frac{(T_S - T_0)\,e^{-C^2}}{\sqrt{\pi}\, C \operatorname{erfc} C} = \text{const}\right) \tag{1.9}$$

Normally the steady-state burning rate increases with increase in pressure; accordingly, the similar solution is realized at decreasing pressures.

2. If the steady-state burning rate is taken in the form (1.1), then (1.9) takes the specific form

$$\frac{C}{\sqrt{\tau}} = \frac{B}{U} p^{\nu} e^{\beta T_*} \tag{2.1}$$

As the characteristic burning rate we take $U = Bp_0^{\nu} e^{\beta T_0}$, where p_0 is the unit of measurement of pressure in (1.1). Then from (2.1) we have

$$\frac{C}{\sqrt{\tau}} = z^{\nu} \exp k\left(1 - \frac{e^{-C^2}}{\sqrt{\pi}\, C \operatorname{erfc} C}\right), \quad z = \frac{p}{p_0}, \quad k = \beta(T_S - T_0) \leqslant 1 \tag{2.2}$$

Here, k is the stability criterion of the steady-state burning regime [1]. From (2.2) we obtain the dimensionless form of the law of pressure variation

$$z = D^{1/\nu} / \tau^{\frac{1}{2\nu}} \tag{2.3}$$

and the relation between the constants C and D in (1.5) and (2.3)

$$\frac{C}{D} = \exp\left[k\left(1 - \frac{e^{-C^2}}{\sqrt{\pi}\, C \operatorname{erfc} C}\right)\right] \tag{2.4}$$

We assume that the constant D in (2.3) is given. Relation (2.4) then defines the burning rate constants C in terms of D. We represent solution (2.5) in graphic form. In Fig. 1 we have plotted the left (straight lines) and right sides of Eq. (2.4) as functions of C at a single value of the parameter k (k = 0.8) and several values of the constant D (D = 1.33, 0.91, and 0.75). At large D Eq. (2.4) has two solutions for the unknown C, one of which corresponds to weak nonstationarity of the burning rate [large value of C; in this case, for the absolute value of the burning rate to change appreciably, a large time is required — see (1.5)], while the other corresponds to strong nonstationarity (small C). At $D < D_*$ a solution of (2.4) does not exist. The values of D_* and the corresponding C_* are found from the condition of tangency of the curves in Fig. 1

$$C_* = \frac{ke^{-C_*^2}}{\sqrt{\pi}\,\mathrm{erfc}\,C_*}\left(1 + 2C_*^2 - \frac{2C_* e^{-C_*^2}}{\sqrt{\pi}\,\mathrm{erfc}\,C_*}\right) \tag{2.5}$$

(The value of D_* is then found from (2.4) by substituting C_*.)

The dependence (2.5) is represented in Fig. 2. As $k \to 0$, which corresponds [1] to a fully preheated and hence perfectly stable propellant, $C_* \to 0$. Thus, of the two solutions, only that corresponding to weak nonstationarity remains. As k tends to the limit of stable steady-state burning ($k \to 1$) $C_* \to C_0 = 0.540$. In section 4 below it is shown that of the two solutions only that which corresponds to the upper intersection of the curves in Fig. 1 is stable.

The absence of a solution at small D can be interpreted, following the ideas expressed in [1], as the quenching of the propellant in the presence of a sharp pressure drop. In this case the quantity D_* can be described as the critical pressure drop parameter characterizing the cessation of combustion (it is shown in Fig. 3 as a function of k). This is a rather conditional interpretation, since the absence of a similar solution does not necessarily imply that there is no solution of the problem at all.

The existence of two similar solutions is not mentioned in [1], where the critical rate of fall of pressure was differently determined – in relation to the attainment of a critical temperature gradient at the surface of the propellant, which leads to an expression for C_* different from (2.5).

Nonuniqueness of the solution is a rather general property of the problem in question for a broad class of functions $u^\circ(p, T_0)$. In particular, it is observed for dependences of the type

$$u^\circ = f_1(p)\, f_2(T_0)$$

where $f_2(T_0)$ – a smooth continuous function – vanishes at a certain value of the argument different from $-\infty$ as T_0 tends from $T_0 = T_S$ to $-\infty$. Examples of such functions are

$$f_2(T_0) = 1 + \alpha T_0, \qquad \frac{1+\alpha T_0}{1-\beta T_0} \quad \text{at} \quad \alpha, \beta > 0, \quad \alpha > \beta$$

and others used in propellant burning theory.

3. We will now find the similar solution of the problem for a propellant model with variable surface temperature and obtain the necessary laws of pressure variation with time and the form of the relations between surface temperature, pressure, and initial temperature admitting similar solutions.

We start from the integral equation obtained in [2], which relates the burning rate, the temperature gradient at the surface, and the surface temperature. In dimensionless form it is written

$$\theta_S(\tau) = \frac{1}{\sqrt{\pi}}\left\{\int_0^\tau \theta_S \exp\left[-\frac{I^2}{4(\tau-\tau')}\right]\left[\frac{\varphi}{\theta_S} - w + \frac{I}{2(\tau-\tau')}\right]\frac{d\tau'}{\sqrt{\tau-\tau'}} + \frac{1}{\sqrt{\tau}}\int_\infty^0 \theta_S(\zeta)\exp\left[-\frac{(\zeta+K)}{4\tau}\right]d\zeta\right\} \tag{3.1}$$

$$I = \int_{\tau'}^{-\infty} w(\tau'')\,d\tau'', \quad K = \int_0^\tau w(\tau'')\,d\tau''$$

where $\theta_0(\xi) = \theta(\xi, 0)$ is the initial steady-state temperature distribution in the subsurface heating zone (before the transient process begins).

We will find the similar solution in the form in which the burning rate is given by relation (1.5), the surface temperature by

$$\theta_S = F\tau^n, \qquad F = \text{const} > 0 \tag{3.2}$$

(n is an arbitrary real number), and the relation between the temperature gradient and the surface temperature by

$$\frac{\varphi}{\theta_S} = \frac{G}{\sqrt{\tau}}, \qquad G = \text{const} > 0 \tag{3.3}$$

Bearing in mind that the similar solution corresponds to an intermediate stage in which the initial conditions are no longer influential (large times τ), we omit the second integral in (3.1) (it tends to zero as $\tau \to \infty$). Then substituting (3.2) and (3.3) in integral equation (1.5) and going over to the new independent variable $\sigma = \tau / \tau'$ in the integrand, we obtain

$$\sqrt{\pi} = \int_0^1 \sigma^n \left(\frac{G}{\sqrt{\sigma}} - \frac{C}{\sqrt{\sigma}} + \frac{C}{1 + \sqrt{\sigma}} \right) \exp \left[- C^2 \frac{1 - \sqrt{\sigma}}{1 + \sqrt{\sigma}} \right] \frac{d\sigma}{\sqrt{1 - \sigma}} \tag{3.4}$$

Thus, the time τ is completely eliminated from the integral equation, which indicates a correct choice of the time relations (1.5), (3.2), and (3.3). Equation (3.4) serves for determining the relation between the constants C and G.

In the particular case $n = 0$, corresponding to the model with $T_S = \mathrm{const}$, the integrals in (3.4) can be evaluated, which leads to a relation between G and C consistent with the results of section 2. At $n \neq 0$ it is necessary to employ numerical methods of calculating the relation G(C) given by Eq. (3.4).

We now turn to the pressure variation that ensures a similar solution of the problem for the model with variable surface temperature. Instead of (1.9) we have

$$\frac{C}{\sqrt{\tau}} = w \left[p, (\tau), \quad T_0 + F \left(1 - \frac{G}{C} \right) (T_S{}^\circ - T_0)\, \tau^n \right] \tag{3.5}$$

In particular, for a dependence of the type (1.1) the pressure should vary as

$$z = \frac{C^{1/\nu}}{\tau^{1/2\nu}} \exp \left[- \frac{kF}{\nu} \left(1 - \frac{G}{C} \right) \tau^n \right] \tag{3.6}$$

where $T_S{}^\circ - T_0$ is the characteristic temperature difference used for normalization, for example, the difference between the surface temperature and the initial temperature in the steady-state burning regime (see below).

We will now consider the conditions that must be satisfied by the dependence of the surface temperature on the initial temperature and pressure in the steady-state regime, for which under nonsteady conditions the surface temperature varies in accordance with (3.2). The transition from the steady-state dependence $\theta_S = \theta_S\,(p,\ T_0)$ to the nonsteady relation between surface temperature and pressure and temperature gradient is made in the same way (see [4]) as the corresponding transformation for the burning rate (1.9)

$$\theta_S = \theta_S \left[p, \quad T_S - \frac{\varkappa}{u} \left(\frac{\partial T}{\partial x} \right)_S \right] \tag{3.7}$$

Using relations (3.2), (1.5), and (3.3), we arrive at the conclusion that the following relation must be identically satisfied:

$$F \tau^n = \theta_S \left[p\,(\tau), \quad T_0 + F \left(1 - \frac{G}{C} \right) (T_S{}^\circ - T_0)\, \tau^n \right] \tag{3.8}$$

where $p(\tau)$ is given by (3.6). Consequently, in order for there to be a similar solution, the function $\theta_S{}^\circ$ $(p,\ T_0)$ must have a perfectly definite form.

In analyzing experimental data on the variation of the surface temperature of a burning propellant it is customary to employ the relation [5] $u = u(p,\ T_S)$, which can be obtained from the relations $u(p,\ \varphi)$ and $T_S(p,\ \varphi)$ by eliminating the gradient. An expression in common use, for example, is the extrapolation formula

$$u = H \exp\,(- E \,/\, R T_S) \tag{3.9}$$

where H is a certain constant (zero-order gasification reaction), R is the gas constant, and E is a characteristic energy, which may be regarded as the activation energy, if it is assumed that gasification is the result of a heterogeneous reaction at the surface of the propellant, or as approximately half the activation energy of the volume decomposition reaction.

If as the characteristic rate we take the steady-state burning rate

$$u = H \exp\,(- E \,/\, R T_S{}^\circ) \tag{3.10}$$

then in dimensionless form (3.9) may be written

$$w = \exp \left(\varepsilon \Delta \, \frac{\theta_S - 1}{1 + \theta_S \Delta} \right)$$

$$\theta_S = \frac{T_S - T_0}{T_S^\circ - T_0}, \quad \varepsilon = \frac{E}{RT_S^\circ}, \quad \Delta = \frac{T_S^\circ - T_0}{T_0} \tag{3.11}$$

Assuming that $|\theta_S - 1| \ll 1$, we can rewrite (3.11) in the approximate form

$$w \cong \theta_S^m \qquad (m = \varepsilon\Delta / (1+\Delta)) \tag{3.12}$$

In the similar solution the burning rate w and the surface temperature θ_S must depend on time in accordance with (1.5) and (3.2). Substituting these functions in (3.12), we see that relation (3.9) (naturally, in the approximation used in deriving (3.12)) admits a similar solution of the problem if as F and n we take

$$F = C^{1/m}, \qquad n = -{}^1/_2 m < 0 \tag{3.13}$$

i.e., if the surface temperature falls with time.

In view of the usually large energy $E|n| \ll 1$. The similar solution of the problem with constant surface temperature may thus be regarded as that corresponding to an infinitely large value of E (n = 0).

The temperature distribution in the propellant can be obtained from the solution of heat conduction equation (1.3). It is reduced to an equation in ordinary derivatives by the choice of similar variable y and the introduction of the new function

$$\theta = \tau^n \vartheta(y), \; y = \xi / 2\sqrt{\tau} \tag{3.14}$$

We have

$$\frac{d^2\vartheta}{dy^2} + 2(y - C)\frac{d\vartheta}{dy} - 4n\vartheta = 0 \tag{3.15}$$

with boundary conditions

$$y = -\infty, \; \vartheta = 0; \; y = 0, \; \vartheta = F = \text{const} \tag{3.16}$$

Equation (3.15) is reduced to a particular form of the Whittaker equation, whose solution is expressed in terms of the Whittaker function [6]

$$\vartheta = \frac{\exp[-{}^1/_2(y-C)^2]}{\sqrt{y-C}}\Big\{C_1 W\Big(\Big(n+\frac{1}{4}\Big), \frac{1}{4}, -(y-C)^2\Big) + C_2 W\Big(-\Big(n+\frac{1}{4}\Big), \frac{1}{4}, (y-C)^2\Big)\Big\} \tag{3.17}$$

where W is the Whittaker function, and C_1 and C_2 are constants of integration, which must be determined from conditions (3.16). At n > 0 the first of the functions W in braces increases without bound as $y \to -\infty$ and, accordingly, the first of conditions (3.16) requires that $C_1 = 0$. The second constant is expressed in terms of F

$$C_2 = \frac{i\sqrt{C}}{\exp(-{}^1/_2 C^2)} W^{-1}\Big(-\Big(n+\frac{1}{4}\Big), \frac{1}{4}, C^2\Big) \qquad (F^m = C) \tag{3.18}$$

The class of similar solutions for the model with variable surface temperature considered above can be further extended by the following simple means. Previously, in solving the heat conduction equation, the space variable ξ was assumed to lie on the interval $[-\infty, 0]$. We now continue the solutions (1.6) and (3.17) onto the interval of the variable $-\infty < \xi < +\infty$ and locate the surface of the propellant at the point $\xi = \xi_0$. At $\xi_0 = 0$ solution (1.6) corresponds to the model with constant surface temperature, and solution (3.17) to that with surface temperature varying according to power law (3.2). When $\xi_0 \neq 0$ the surface temperature is a function of time that is neither constant nor the same as (3.2). In particular, at n = 0 this function takes the form

$$T_S(\xi_0, \tau) = T_0 + \frac{T_S^\circ - T_0}{\operatorname{erfc} C}\Big[1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta'} e^{-\eta^2} d\eta\Big]\Big(\eta' = C - \frac{\xi_0}{2\sqrt{\tau}}\Big) \tag{3.19}$$

If $\xi_0 < 0$, then the surface temperature increases with time and at large times tends in the limit to the constant value T_S°. If, however, $\xi_0 > 0$, then the surface temperature decreases with time and likewise tends to T_S°.

Calculating the temperature gradient at the surface from (3.19) and substituting, as before, in (1.9), we obtain a relation giving the required law of pressure variation

$$\frac{C}{\sqrt{\tau}} = w\left[p(\tau),\ \ T_S{}^{\circ} - \frac{T_S{}^{\circ} - T_0}{C\sqrt{\pi}\,\mathrm{erfc}\,C}\exp\left(-\left(C - \frac{\xi_0}{2\sqrt{\tau}}\right)^2\right)\right] \tag{3.20}$$

From (3.9) there follows a relation for the surface temperature that defines the class of propellants with variable surface temperature for which it is possible to obtain similar solutions:

$$\theta_S = \theta_S\left[p(\tau),\ \ T_S{}^{\circ} - \frac{T_S{}^{\circ} - T_0}{C\sqrt{\pi}\,\mathrm{erfc}\,C}\exp\left(-\left(C - \frac{\xi_0}{2\sqrt{\tau}}\right)^2\right)\right] \tag{3.21}$$

We note that (3.21) contains the free parameter ξ_0, which can be used, together with $T_S{}^{\circ}$, to approximate the experimental data. A similar procedure can also be followed when $n \neq 0$.

4. In Sec. 2 we showed in relation to the model with constant surface temperature that there are two solutions of the similar problem. The question arises, which of these solutions is actually realized, i.e., which is stable.

As distinct from investigations of the stability of steady-state solutions, where those solutions for which the absolute value of infinitely small perturbations increases with time are considered unstable, in the case of similar solutions it is necessary to examine the behavior of the relative value of the perturbations – the ratio of the perturbation amplitude to the undisturbed solution, which is also a function of time. (This approach was previously used to investigate the stability of a laminar spherical flame in [7].) In this case solutions whose trajectory in the presence of perturbations remains close to that without perturbations are considered stable.

If in heat conduction equation (1.3) we go over from the independent variables (τ, ξ) to the variables $(\zeta = \ln \tau, y)$ and from the function θ to the function ϑ in accordance with (3.14), then after linearization we obtain an equation for the perturbations that does not contain the variable ζ in explicit form

$$4\frac{\partial\delta\vartheta}{\partial\zeta} = \frac{\partial^2\delta\vartheta}{\partial y^2} + 2(y - C)\frac{\partial\delta\vartheta}{\partial y} - 2C\delta\Sigma\frac{d\vartheta^{\circ}}{dy} - 4n\delta\vartheta \tag{4.1}$$

Here, δ denotes the small perturbations, $\delta\Sigma$ is the relative value of the burning rate perturbation $\delta\Sigma = \delta w/w$, ϑ° is the undisturbed solution (3.17).

As usual, the solution of Eq. (4.1) can be found in exponential form for ζ. However, it is also necessary that the additional relations for the perturbation of burning rate, surface temperature, and temperature gradient, which follow from the algebraic relations between the burning rate, pressure, and temperature gradient and between the surface temperature, pressure, and temperature gradient, do not contain time in explicit form.

It turns out that this is possible only for a specific form of the relation between the steady-state burning rate, pressure, and initial temperature. In fact, the function (3.5) can be written in the form

$$w = \frac{C}{\sqrt{\tau}} = \Psi(z, j), \qquad j = \theta_S - \frac{\varphi}{w} \tag{4.2}$$

Linearizing (4.2) and using (3.2), (3.3),

$$\frac{\delta w}{w} = -\frac{\Psi_j}{\Psi}\frac{G}{C}F\tau^n\left[1 - \frac{\Psi_j}{\Psi}F\tau^n\left(\frac{1}{m} + \frac{G}{C}\right)\right]^{-1} \qquad \Psi_j = \frac{\partial\Psi}{\partial j} \tag{4.3}$$

In order for it to be possible to employ the method of solution using exponential perturbation relations in $\zeta = \ln \tau$, it is necessary that

$$\frac{\Psi_j}{\Psi}F\tau^n = \frac{\Psi_j}{\Psi}F\left(\frac{C}{\Psi}\right)^{2n} = \mathrm{const} \tag{4.4}$$

Integrating this relation, we find

$$\Psi = \left[Z(z) - \mathrm{const}\left(\theta_S - \frac{\varphi}{w}\right)\right]^{-1/2n}, \qquad n \neq 0 \tag{4.5}$$

$$\Psi = Z(z)\exp\left[\mathrm{const}\left(1 - \frac{\varphi}{w}\right)\right], \qquad n = 0 \tag{4.6}$$

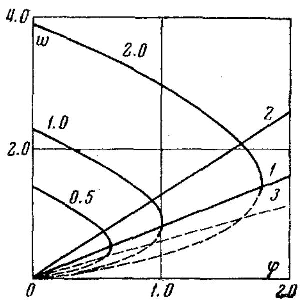

Fig. 4

where Z(z) is an arbitrary function of the dimensionless pressure z. We note, in particular, that the previously considered burning law (1.1) leads to a functional dependence of the type (4.6).

If the solutions for the perturbations are represented in the form of exponentials in log time

$$\delta\vartheta = f^{(1)}(y)\, e^{\omega\zeta}, \qquad \delta\Sigma = \mathrm{const}\, e^{\omega\zeta} \tag{4.7}$$

then the solution investigated for stability is assumed to be unstable at Re $\omega > 0$. If we return to the starting variable, time τ, then (4.7) corresponds to the power law

$$\delta\vartheta \sim \tau^{\omega}, \ \delta w = w\delta\Sigma = \mathrm{const}\, \tau^{\omega - 1/2} \tag{4.8}$$

and on going over to the perturbation $\delta\theta$

$$\delta\theta(\tau, y) = f^{(2)}(y)\, \tau^{n+\omega} \tag{4.9}$$

(the form of the function $f^{(2)}(y)$ must be found from the solution of the heat conduction equation). The perturbation of the temperature gradient at the surface of the propellant is calculated from (4.9) as

$$\delta\varphi = \left(\frac{\partial\delta\theta}{\partial\xi}\right)_{\xi=0} = \frac{1}{2}\tau^{n-1/2+\omega}\left(\frac{df^{(2)}}{dy}\right)_{y=0} \tag{4.10}$$

We now find the limit of stability of the solutions for the model with constant surface temperature (n = 0). For this purpose we employ integral equation (3.1), which, if we separate the perturbations of the temperature gradient $\delta\varphi$, the burning rate δw, and integral δI, can be rewritten for the similar solution in the form

$$\int_0^1 \Big\{\delta\varphi(x\tau) - \delta w(x\tau) + \frac{1}{2(1-x)}\int_x^1 \delta w(\tau s)\, ds\Big[1 + 2C(C-G) \times \left(\frac{1}{\sqrt{x}} - 1\right) - 2C^2\frac{1-\sqrt{x}}{1+\sqrt{x}}\Big]\exp - C^2\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\Big\}\frac{dx}{\sqrt{1-x}} = 0 \tag{4.11}$$

Here, we have introduced the variable $x = \tau'/\tau$ and used the form of the undisturbed solution (1.5).

From the relation obtained it is clear that in accordance with (4.8) and (4.10) the integral does not depend on time τ, if the perturbations of the burning rate and temperature gradient are represented in the form

$$\delta\varphi(\tau) = \delta\varphi_0 \tau^{l}, \ \delta w = \delta w_0 \tau^{l} \tag{4.12}$$

where l is an arbitrary real number; comparison with (4.8) gives $l = \omega - 1/2$. Substituting (4.12) in (4.11), we obtain a relation between the amplitudes $\delta\varphi_0$ and δw_0. In what follows we shall be interested only in the stability limit for which Re $\omega = 0$. We further assume that at the limit Im $\omega = 0$, calculations show that the relations obtained can be satisfied on this assumption, which also points to its validity. Evaluation of the integrals leads to a relation between the gradient and burning rate perturbations

$$\frac{\delta\varphi_0}{\delta w_0}\pi e^{-C^2}(1 - \mathrm{erf}^2 C) = \frac{2e^{-C^2}(1 + \mathrm{erf}\, C)}{\mathrm{erfc}\, C} - 2\sqrt{\pi}C(1 + \mathrm{erf}\, C) \tag{4.13}$$

We obtain a second relation between $\delta\varphi_0$ and δw_0 by considering the function relating the burning rate, pressure, and temperature gradient. From the steady-state burning law (1.1) we easily obtain

$$\frac{\delta\varphi_0}{\delta w_0} = \frac{1}{k}\left(k\frac{G}{C} - 1\right) \tag{4.14}$$

(in the variation process it is assumed that z = const).

From (4.13) and (4.14) we obtain a relation between C and k at the stability limit which has a form that coincides identically with tangency condition (2.5). The point of tangency is the critical point separating the stable from the unstable solutions of the problem. At this point both the real and imaginary parts of the characteristic frequency ω vanish.

Bearing in mind that in the limit at large C the upper point of intersection of the curves in Fig. 1 describes burning with weak nonstationarity, from continuity considerations we conclude that it corresponds to the stable solution, and the lower point of intersection to the unstable solution.

We represent the results obtained in a graph in which the burning rate is plotted against the temperature gradient at the surface (Fig. 4). To be specific, we assume that the dependence of the steady-state burning rate on pressure and initial temperature is given by Eq. (1.1), which, on going over to nonsteady burning conditions, yields the relation

$$w = z^{\nu} \exp k\left(1 - \frac{\varphi}{w}\right) \tag{4.15}$$

In Fig. 4 the relation $w(\varphi)$ (4.15) has been plotted for three values of the pressure ($z^{\nu} = 0.5$, 1.0, 2.0). The curves have infinite derivatives, which, as it is easy to see, lie on the straight line $w = k\varphi$ (straight line 1).

Steady-state burning at various pressures and a single value of the initial temperature (which was selected as the characteristic temperature) is described by the straight line $w = \varphi$ (straight line 2).

As the initial temperature varies, so does the slope of the straight line corresponding to steady-state burning. In particular, the slope of the straight line (1.1) corresponds to the temperature $T_0{}^*$ at which $\beta\,(T_S - T_0{}^*) = 1$, i.e., corresponds to the stability limit of the steady-state solutions. Accordingly, the lower unstable branches of the $w(\varphi)$ curves in Fig. 4 are shown dashed.

In Fig. 4 the similar solutions are also represented by straight lines. In fact, from the relations of Sec. 2 we have

$$w = \sqrt{\pi}\, C e^{C^2} \operatorname{erfc} C \varphi \tag{4.16}$$

At large C straight line (4.16) tends to straight line 2 corresponding to steady-state burning (weak nonstationarity). As $C \to 0$ (strong nonstationarity) straight line (4.16) approaches the axis of abscissas and enters the region of unstable steady-state solutions.

The stability limit of the self-similar regimes is obtained by substituting in (4.16) the critical value C_* given by (2.5). It has also been plotted in Fig. 4 (dashed straight line). Clearly, the region of stability of the similar solutions is broader than the region of stability of the steady-state solutions; this is associated with the different definition of an unstable perturbation.

We note that in [1] the criterion for determining the stability limit of the similar solutions was the attainment of the critical temperature gradient at the surface of the propellant. In this case no consideration was given to the question of how the perturbation varies in time as compared with the variation of the undisturbed quantity.

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